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Interpolation Mixed with l_2-Approximation

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We consider sqn roots of unity and define a class $\mathscr{R}_{N^*}^q(U_s, f)$ of rational functions which interpolate a given analytic function f on U_s, a large subset of the roots of unity satisfying a congruence relation. f is then approximated over $\mathscr{R}_{N^*}^q(f, U_s)$ with respect to l_2-norm on the complement of U_s. We also discuss Walsh type equiconvergence.

1. INTRODUCTION

Let π_s denote the family of all polynomials of degree $\leqslant s$, and let $L_{n-1}(z, f) \in \pi_{n-1}$ denote the Lagrange interpolant to a function f analytic in the region $|z| < \eta$, $\eta > 1$, at the n roots of $z^n = 1$. It is well-known [7] that the best l_2-approximant to f from π_{n-1} over the set of the n zeros of $z^n - 1$ is $L_{n-1}(z, f)$. During the last decade several papers have appeared on discrete least squares minimization problems considered over a large set of the primitive roots of unity. In [3] Rivlin noted that the $(n-1)$th degree polynomial which solves the problem

$$\min_{p \in \pi_{n-1}} \sum_{k=0}^{qn-1} |f(\omega^k) - p(\omega^k)|^2, \qquad \omega^{qn} = 1, q \geqslant 1 \tag{1.1}$$

is essentially $S_{n-1}[z, L_{qn-1}(z, f)]$, the $(n-1)$th degree Taylor section of the polynomial $L_{qn-1}(z, f)$. In a different direction Sharma and Ziegler considered the following question [4]: If $\mathscr{L}(f, U_s)$ denotes the class of all polynomials of degree $\leqslant nq(s-1) + n - 1$ interpolating f on the set

$$U_s = \{\omega^v : v = 1, 2, ..., sqn;\ v \not\equiv 0 \pmod{s};\ \omega^{sqn} = 1\}, \tag{1.2}$$

find the solution to the problem

$$\min_{Q \in \mathscr{L}(f, U_s)} \sum_{v=0}^{qn-1} |f(\lambda^v) - Q(\lambda^v)|^2, \qquad \lambda^{sqn} = 1, \lambda \notin U_s. \tag{1.3}$$

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They discovered that the solution $Q_n^*(z, f) \in \mathscr{L}(f, U_s)$ to (1.3) is given by

$$Q_n^*(z, f) = L^*(z, f) + W_s(z)\, S_{n-1}[z, L_{qn-1}(z, g)] \tag{1.4}$$

where $L^*(z, f)$ is the Lagrange interpolant of degree $nq(s-1)-1$ to f on U_s, $g(z) := s^{-1}[f(z) - L^*(z, f)]$ and

$$W_s(z) = (z^{sqn} - 1)/(z^{qn} - 1). \tag{1.5}$$

The aim of the present note is two fold. First, we develop a variant of the minimization problem (1.3) replacing $\mathscr{L}(f, U_s)$ by a class of certain interpolatory rational functions. The second problem to be discussed here is related to Walsh-type equiconvergence. This topic has attracted many mathematicians in the last decade. For the background we refer the reader to [2]–[6].

2. Preliminaries and Statement of Problems

We denote by A_ρ, $1 < \rho < \infty$, the class of functions analytic in $|z| < \rho$ with at least one singularity on $|z| = \rho$, and set

$$N = qn(s-1) \qquad \text{and} \qquad N^* = N + n + m \tag{2.1}$$

where $s \geqslant 1$, $q \geqslant -1$ are fixed integers. For a given $\sigma > 1$, let $\mathscr{R}^\sigma_{v,n}$ denote the class of rational functions $r(z)$ of the form

$$r(z) = p(z)/(z^n - \sigma^n), \qquad p \in \pi_v. \tag{2.2}$$

With the set U_s defined in (1.2) and an $f \in A_\rho$, let $\mathscr{R}^\sigma_{v,n}(f, U_s)$ denote the subclass of rational functions $r \in \mathscr{R}^\sigma_{v,n}$ which interpolate f on U_s.

We shall consider the following problems:

(P_1) For a given $f \in A_\rho$, $\rho > 1$, find the rational function $R^*_{N^*,n}(z, f) \in \mathscr{R}^\sigma_{N^*,n}(f, U_s)$ which solves the problem

$$\min_{R \in \mathscr{R}^\sigma_{N^*,n}(f, U_s)} \sum_{v=0}^{qn-1} |f(\lambda^v) - R(\lambda^v)|^2, \qquad \lambda^{sqn} = 1, \lambda \notin U_s \tag{2.3}$$

(P_2) If $r_{N^*,n}(z, f) \in \mathscr{R}^\sigma_{N^*,n}$ minimizes $f \in A_\rho$ on $|z| = 1$ in the L_2-sense over the class $\mathscr{R}^\sigma_{N^*,n}$ ([5], (1.4)) and if $R^*_{N^*,n}(z, f)$ is the solution of (P_1), what is the region of convergence of the difference

$$R^*_{N^*,n}(z, f) - r_{N^*,n}(z, f) \tag{2.4}$$

to zero as $n \to \infty$?

Remark 2.1. When $s = 1$, the solutions to the problems (P_1) and (P_2) are provided in [1]. For the justification, it is enough to note that the set U_s which consists of the zeros of $W_s(z)$ is empty for $s = 1$.

3. Solution of (P_1)

In order to solve the problem (P_1), we need an expression for the Lagrange interpolant of the function

$$h(z) := s^{-1}[f(z) - R^*_{N-1,n}(z, f)] \tag{3.1}$$

where $R^*_{N-1,n}(z, f)$ is the rational function in $\mathscr{R}^q_{N-1,n}$ which interpolates f on U_s. More precisely, we need the following Lemma:

Lemma 3.1. *If $s \geqslant 2$ and $q \geqslant 1$ are fixed integers and if*

$$L_{nq-1}(z, h) = \sum_{j=0}^{q-1} \sum_{v=0}^{n-1} c_{v+jn} z^{v+jn} \tag{3.2}$$

is the Lagrange polynomial of degree $qn - 1$ which interpolates the function $h(z)$ at the roots of $z^{qn} = 1$, then

$$c_{v+jn} = \frac{1}{2\pi i} \int_\Gamma \left[\frac{\lambda_0(q) t^{n-v-1}}{\sigma^{(j+1)n}} + \frac{t^{(q-j)n-v-1}}{t^{qn} - 1} \right] \frac{f(t)}{W_s(t)} dt, \tag{3.3}$$

where Γ is a circle $|t| = \eta$, $1 < \eta < p$ and

$$\lambda_0(q) := \sigma^{qn}/(1 - \sigma^{qn}). \tag{3.4}$$

Proof. In order to establish (3.4), we first find $L_{nq-1}(z, R^*_{N-1,n}(z, f))$. Since

$$R^*_{N-1,n}(z, f) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n)}{W_s(t)} \cdot \frac{G(t, z)}{z^n - \sigma^n} f(t)\, dt. \tag{3.5}$$

where

$$G(t, z) := \frac{W_s(t) - W_s(z)}{t - z},$$

it is enough to evaluate $L_{qn-1}(z, G(t, z)/z^n - \sigma^n)$. It is easy to see that

$$L_{qn-1}(z, G(t, z)) = \sum_{\nu=0}^{s-1} L_{qn-1}\left(z, \frac{t^{\nu qn} - z^{\nu qn}}{t-z}\right) = \frac{t^{qn} - z^{qn}}{t-z} \sum_{\nu=0}^{s-1} \frac{t^{\nu qn} - 1}{t^{qn} - 1}$$

$$= \frac{t^{qn} - z^{qn}}{t-z}\left[\frac{t^{sqn} - 1}{(t^{qn} - 1)^2} - \frac{s}{t^{qn} - 1}\right]$$

$$= \frac{t^{qn} - z^{qn}}{t-z}\left[\frac{W_s(t)}{t^{qn} - 1} - \frac{s}{t^{qn} - 1}\right].$$

Therefore,

$$L_{qn-1}\left(z, \frac{G(t, z)}{z^{n-\sigma}_{\ \ n}}\right) = S_1(t, z) - S_2(t, z)$$

where we have set

$$\begin{aligned} S_1(t, z) &= \frac{W_s(t)}{t^{qn} - 1} L_{qn-1}\left(z, \frac{t^{qn} - z^{qn}}{t-z} \cdot \frac{1}{z^n - \sigma^n}\right) \\ S_2(t, z) &= \frac{s}{t^{qn} - 1} L_{qn-1}\left(z, \frac{t^{qn} - z^{qn}}{t-z} \cdot \frac{1}{z^n - \sigma^n}\right). \end{aligned} \tag{3.6}$$

It follows from (3.5) that

$$L_{qn-1}(z, R^*_{N-1,n}(z, f)) = I_1(z) - I_2(z) \tag{3.7}$$

where

$$I_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n) f(t)}{W_s(t)} S_1(t, z)\, dt,$$

and

$$I_2(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n) f(t)}{W_s(t)} S_2(t, z)\, dt.$$

Since

$$\frac{(t^{qn} - z^{qn})(t^n - \sigma^n)}{(t-z)(z^n - \sigma^n)} = \frac{t^{qn} - z^{qn}}{t-z} + \frac{(t^n - z^n)(t^{qn} - z^{qn})}{(t-z)(z^n - \sigma^n)},$$

we have

$$I_1(z)=\frac{1}{2\pi i}\int_\Gamma \frac{(f(t)(t^{qn}-z^{qn})}{(t-z)(t^{qn}-1)}\,dt$$

$$+\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{qn}-1}\,L_{qn-1}\left(z,\frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)}\right)dt. \tag{3.8}$$

The second integral in (3.8) vanishes because

$$\frac{f(t)}{t^{qn}-1}\,L_{qn-1}\left(z,\frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)}\right)=\frac{f(t)(t^n-z^n)}{(t-z)(z^n-\sigma^n)}.$$

Therefore,

$$I_1(z)=L_{qn-1}(z,f). \tag{3.9}$$

Similarly,

$$I_2(z)=I_3(z)+I_4(z) \tag{3.10}$$

where

$$I_3(z)=\frac{s}{2\pi i}\int_\Gamma \frac{f(t)}{t^{sqn}-1}\,L_{qm-1}\left(z,\frac{t^{qn}-z^{qn}}{t-z}\right)dt$$

$$=\frac{s}{2\pi i}\sum_{v=0}^{qn-1} z^v\int_\Gamma \frac{f(t)}{t^{v+1}}\cdot\frac{t^{qn}}{t^{sqn}-1}\,dt, \tag{3.11}$$

and

$$I_4(z)=\frac{s}{2\pi i}\int_\Gamma \frac{f(t)}{t^{sqn}-1}\,L_{qn-1}\left(z,\frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)}\right)dt$$

Since $L_{qn-1}(z, 1/z^n-\sigma^n)=(\sigma^{qn-n}/1-\sigma^{qn})\sum_{j=0}^{q-1} z^{jn}/\sigma^{jn}$, it is easy to see that

$$I_4(z)=\frac{s}{2\pi i}\lambda_0(q)\int_\Gamma \frac{f(t)}{W_s(t)}\sum_{j=0}^{q-1}\sum_{v=0}^{n-1}\frac{t^{n-v-1}z^{jn+v}}{\sigma^{nj+n}}\,dt \tag{3.12}$$

where $\lambda_0(q)$ is defined in (3.4). Combining (3.7) with (3.9)–(3.10) we obtain

$$L_{qn-1}(z, R^*_{N-1,n}(z,f))=L_{qn-1}(z,f)-I_3(z)-I_4(z).$$

Using (3.1), we observe that

$$L_{qn-1}(z,h)=s^{-1}[L_{qn-1}(z,f)-L_{qn-1}(z,R^*_{N-1,n}(z,f))]$$
$$=s^{-1}[I_3(z)+I_4(z)].$$

Finally, the substitution of the values of $I_3(z)$ and $I_4(z)$ from (3.11)-(3.12) in the above relation establishes the equation (3.2) for which the coefficients are given in (3.3). ∎

Now we proceed to determine a solution of the problem (P_1). First we note that any rational function $R(z)\in\mathscr{R}^{\sigma}_{N^*,n}(f,U_s)$ can be expressed as

$$R(z)=R^*_{N-1,n}(z,f)+W_s(z)\,B(z)$$

for some $B(z)\in\mathscr{R}^{\sigma}_{n+m,n}$. Here $R^*_{N-1,n}(z,f)$ is the rational function used in (3.1). Since $W_s(\lambda^{\nu})=s$ for any solution λ of $z^{sqn}=1$, it follows that

$$|f(\lambda^{\nu})-R(\lambda^{\nu})|^2=|f(\lambda^{\nu})-R^*_{N-1,n}(\lambda^{\nu},f)-sB(\lambda^{\nu})|^2$$
$$=|sh(\lambda^{\nu})-sB(\lambda^{\nu})|^2.$$

Thus the problem (P_1) is equivalent to minimizing

$$\sum_{\nu=0}^{qn-1}|h(\omega^{\nu})-B(\omega^{\nu})|^2,\qquad \omega^{qn}=1, \tag{3.13}$$

over all rational functions $B\in\mathscr{R}^{\sigma}_{n+m,n}$. This problem had been solved by the author in [1]. In fact, if $L_{qn-1}(z,h):=\sum_{\nu=0}^{qn-1}c_{\nu}z^{\nu}$ then ([1], Proposition 1)

$$B^*_{n+m,n}(z,h):=\sum_{\nu=0}^{n+m}\tau_{\nu}z^{\nu}/(z^n-\sigma^n) \tag{3.14}$$

will be the minimizer of (3.13) over $\mathscr{R}^{\sigma}_{n+m,n}$ where

$$\tau_{\nu}=\begin{cases}-c_{\nu}\sigma^n+\dfrac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}}\displaystyle\sum_{j=1}^{q-1}\sigma^{-jn}c_{\nu+jn}, & 0\leqslant\nu\leqslant m\\[2ex] \dfrac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}}\displaystyle\sum_{j=0}^{q-1}\sigma^{-jn}c_{\nu+jn}, & m+1\leqslant\nu\leqslant n-1\\[2ex] c_{\nu-n}-\dfrac{\sigma^{2(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}}\displaystyle\sum_{j=1}^{q-1}\sigma^{-jn}c_{\nu+(j-1)n}, & n\leqslant\nu\leqslant n+m\end{cases} \tag{3.15}$$

Now the description of c_j's given in Lemma 3.1 can be applied to τ_ν's for the explicit representation of $B^*_{n+m,n}(z, h)$. Thus

$$R^*_{N^*,n}(z, f) := R^*_{N-1,n}(z, f) + W_s(z)\, B^*_{n+m,n}(z, h) \qquad (3.16)$$

is the rational function in $\mathscr{R}^{\sigma}_{N^*,n}(f, U_s)$ which provides the desired solution of (P_1).

Remark 3.2. The relation (3.15) reduces to $\tau_\nu = (1-\sigma^n)c_\nu$, $0 \leqslant \nu \leqslant n-1$ when $q=1$ and $m=-1$. Thus, the rational function $B^*_{n-1,n}(z, h)$ turns out to be $[(1-\sigma^n)/(z^n-\sigma^n)]\, L_{n-1}(z, h)$. Consequently, the solution (3.16) to (P_1), in this case, bases entirely on the interpolatory character of the rational functions $R^*_{N-1,n}(z, f)$ and $B^*_{n-1,n}(z, h)$.

4. Solution of (P_2)

The problem (P_2) deals with Walsh-type equiconvergence. Here we shall provide its solution and note that it extends an earlier result due to Sharma and Ziegler ([4], Theorem 1). In order to avoid lengthly expressions in the calculations, we shall discuss the problem (P_2) for $m=-1$. However, the solution stands valid for any integer $m<-1$. More precisely, we prove

Theorem 4.1. *Let $s \geqslant 2$ and $q \geqslant 2$ be fixed integers, and let $N := (s-1)qn$. If $f \in A_\rho$, $1<\rho<\infty$, and $\sigma>1$ then (cf. (2.4))*

$$\lim_{n\to\infty} \{R^*_{N+n-1,n}(z, f) - r_{N+n-1,n}(z, f)\} = 0, \qquad \forall z \in D_\sigma \qquad (4.1)$$

where D_σ for $q \geqslant 3$ is given by

(i)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho^{1+1/(s-1)}\}$	*if* $\sigma \geqslant \rho^{q-1}$
(ii)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho^{1+1/(s-1)q}\cdot\sigma^{1/(s-1)q},\ \lvert z\rvert \neq \sigma\}$	*if* $\rho \leqslant \sigma < \rho^{q-1}$
(iii)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho\cdot\sigma^{2/(s-1)q},\ \lvert z\rvert \neq \sigma\}$	*if* $\sigma<\rho$,

and for $q=2$, D_σ is given by

(i)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho^{1+1/(s-1)}\}$	*if* $\sigma \geqslant \rho^{s}$
(ii)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho^{1+3/(2s-3)}\cdot\sigma^{1/(3-2s)},\ \lvert z\rvert \neq \sigma\}$	*if* $\rho \leqslant \sigma < \rho^{s}$
(iii)	$\{z\in\mathscr{C} : \lvert z\rvert < \rho\cdot\sigma^{1/(s-1)},\ \lvert z\rvert \neq \sigma\}$	*if* $\sigma<\rho$.

Moreover, the convergence is uniform and geometric in any compact subset of region D_σ.

The proof of the above theorem requires integral representations of the rational functions $R^*_{N+n-1,n}(z,f)$ and $r_{N+n-1,n}(z,f)$ together with some estimates which we describe below as remarks.

Remark 4.1. It is known that ([5], (1.4))

$$r_{N+n-1,n}(z,f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{t^n-\sigma^n}{z^n-\sigma^n}\cdot\frac{k(t)-k(z)}{k(t)}\,dt \tag{4.2}$$

where Γ is a circle $|t|=\rho'$, $1<\rho'<\rho$, and

$$k(t)=t^N(t^n-\sigma^{-n}). \tag{4.3}$$

Remark 4.2. If we set

$$\delta_1=\frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}},\qquad \delta_2=\frac{1-(\sigma t)^{-qn}}{1-(\sigma t)^{-n}} \tag{4.4}$$

then using (3.3) we can write

$$\sum_{j=0}^{q-1}\sigma^{-jn}c_{v+jn}=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{t^{v+1}W_s(t)}\left\{\frac{t^n}{\sigma^n}\lambda_0(q)+\frac{t^{qn}}{t^{qn}-1}\delta_2\right\}dt$$

where $\lambda_0(q)$ is given by (3.4). Thus the numerator of the rational function $B^*_{n-1,n}(z,h)$ (cf. (3.14)) can be expressed as

$$\sum_{v=0}^{n+1}\tau_v z^v=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{W_s(t)}\left\{\frac{t^n}{\sigma^n}\delta_1\,\lambda_0(q)+\frac{t^{qn}}{t^{qn}-1}\delta_1\,\delta_2\right\}\frac{t^n-z^n}{t^n(t-z)}\,dt. \tag{4.5}$$

Remark 4.3. Due to the interpolatory properties of the rational function $R^*_{N-1,n}(z,f)$ (cf. (3.1)) we have the following integral representations:

$$R^*_{N-1,n}(z,f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{t^n-\sigma^n}{z^n-\sigma^n}\cdot\frac{W_s(t)-W_s(z)}{W_s(t)}\cdot\frac{f(z)}{t-z}\,dt. \tag{4.6}$$

Remark 4.4. Since $\sigma>1$, we have the following estimates (cf. (4.4), (3.4)):

$$\begin{aligned}\delta_1\lambda_0(q)&=\sigma^n(1-\sigma^{-2n})+O(\sigma^{-2qn+n})\\ \delta_2\lambda_0(q)&=\frac{t^n\sigma^n(\sigma^{-2n}-1)}{t^n-\sigma^{-n}}+O(\sigma^{-qn+n}).\end{aligned} \tag{4.7}$$

The above remarks bring us to the

Proof of Theorem 4.1. From (4.2), (4.5), and (4.6) we can write

$$R^*_{N+n-1}(z,f)-r_{N+n-1}(z,f)$$
$$=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\{K_1,z,\sigma)+K_2(t,z,\sigma)-K_3(t,z,\sigma)\}\,dt \tag{4.8}$$

where

$$K_1(t,z,\sigma)=\frac{k(z)}{k(t)}\cdot\frac{t^n-\sigma^n}{z^n-\sigma^n},$$

$$K_2(t,z,\sigma)=\frac{W_s(z)}{W_s(t)}\cdot\frac{1}{z^n-\sigma^n}\left\{\frac{t^n}{\sigma^n}\delta_1\lambda_0(q)+\frac{t^{qn}}{t^{qn}-1}\delta_1\delta_2-(t^n-\sigma^n)\right\},$$

$$K_3(t,z,\sigma)=\frac{W_s(z)}{W_s(t)}\cdot\frac{1}{z^n-\sigma^n}\left\{\frac{t^n}{\sigma^n}\delta_1\lambda_0(q)+\frac{t^{qn}}{t^{qn}-1}\delta_1\delta_2\right\}\frac{z^n}{t^n}.$$

After the cancellation of suitable terms and using (4.7) together with the estimate $W_s(z)/W_{ss}(t)=O(z^N/t^N)$ we notice that

$$K_2(t,z,\sigma)=\frac{1}{z^{n-\sigma}{}_n}\,O\left(\frac{z^N}{t^N}\max\left\{\frac{|t|^n}{\sigma^{2n}},\frac{\sigma^n}{|t|^{qn}},\frac{1}{\sigma^n}\right\}\right). \tag{4.9}$$

Similarly, a detailed analysis with appropriate cancellation of certain terms of higher order leads us to

$$K_1(t,z,\sigma)-K_3(t,z,\sigma)$$
$$=\frac{1}{z^n-\sigma^n}\left\{O\left(\frac{z^N}{t^N}\max\left\{\frac{1}{\sigma^n},\frac{1}{|t|^n}\right\}\right)\right.$$
$$+O\left(\frac{z^{N+n}}{t^N}\max\left\{\frac{1}{|t|^{sqn}},\frac{1}{|t|^n\sigma^n},\frac{\sigma^n}{|t|^{sqn+n}},\frac{1}{\sigma^{2n}}\right\}\right)$$
$$\left.+O\left(\frac{z^{N-qn+n}}{t^N}\max\left\{1,\frac{\sigma^n}{|t|^n}\right\}\right)\right\}. \tag{4.10}$$

Thus, from (4.9) and (4.10) we can write

$$K_1(t,z,\sigma)+K_2(t,z,\sigma)-K_3(t,z,\sigma)$$
$$=\frac{1}{z^n-\sigma^n}\{O(T_1)+O(T_2)+O(T_3)\}$$

where

$$T_1=\frac{z^{(s-2)qn+n}}{t^{(s-1)qn}}\max\left\{1,\frac{\sigma^n}{|t|^n}\right\},$$

$$T_2=\frac{z^{(s-1)qn}}{t^{sqn}}\max\left\{\frac{|t|^{(q+1)n}}{\sigma^{2n}},\frac{|t|^{qn}}{\sigma^n},|t|^{(q-1)n},\sigma^n\right\},$$

$$T_3=\frac{z^{(s-1)qn+n}}{t^{(s-1)qn}}\max\left\{\frac{1}{|t|^{sqn}},\frac{1}{|t|^n\sigma^n},\frac{\sigma^n}{|t|^{sqn+n}},\frac{1}{\sigma^{2n}}\right\}.$$

After considering various cases for σ separately for the integers $q\geqslant 3$ and $q=2$, we analyze the order of the terms T_1, T_2, T_3. This leads us to the determination of different regions of convergence for (4.8) as desired in the theorem. ∎

Remark 4.5. Theorem 4.1 does not consider the case when $s=1$ or $q=1$. These cases are already settled in [1] and [5], respectively. It is worth mentioning that problem (P_1) entirely deals with l_2-minimization if $s=1$, whereas it reduces to an interpolation problem when $q=1$.

Remark 4.6. IIf we let $\sigma\to\infty$ in Theorem 4.1, we retrive a result of Sharma–Ziegler ([4], Theorem 1).

5. Generalization of Problems P_1 and P_2

The problem P_1 discussed above involves the nodes distributed uniformly on the unit circle. We may formulate this problem in a more general setting where the nodes are selected on the circles of radii α and β with $\max\{\alpha,\beta\}<\rho$. The underlying idea in this set up is due to a result of Lou Yuanren [8]. We conclude our paper with the following generalization of problems P_1 and P_2:

(P_1^*) Let $0<\alpha<\rho$ be a real number. Consider the zeros of $z^{qns}-\alpha^{qns}$ and divide them into two disjoint sets $U_{s,\alpha}$ and $V_{s,\alpha}$ where $V_{s,\alpha}=\{$set of zeros of $z^{qn}-\alpha^{qn}\}$ and $U_{s,\alpha}=\{$set of zeros of polynomial $W_{z,\alpha}(z)=(z^{qns}-\alpha^{qns})/(z^{qn}-\alpha^{qn})\}$. Then $\# V_{s,\alpha}=nq$ and $\# U_{s,\alpha}=N:=qn(s-1)$ where $\# V$ denotes the cardinality of a set V. If $\sigma>1$ is any real number, let R^σ_{N+m+n} denote the class of rational functions of the form $p(z)/(z^n-\sigma^n)$, $p(z)\in\pi_{N+n+m}$. We shall denote by $R^\sigma_{N+m+n,\alpha}(z,f)$ the subset of rational functions from the class R^σ_{N+m+n} which interpolate

$f(z) \in A_\rho$, $\rho > 1$, on the set $U_{s,\alpha}$. The problem is to determine the rational functions $R(z, \alpha, f) \in R^{\sigma}_{N+m+n,\alpha}$ which minimizes.

$$\sum_{k=0}^{qn-1} |f(\alpha\omega^k) - R(\alpha\omega^k, f)|^2, \qquad \omega^{qn} = 1. \tag{5.1}$$

(P_1^{**}) In this problem, we replace the set $V_{s,\alpha}$ by $V_{s,\beta}$ where $0 < \alpha \neq \beta < \rho$ and seek the rational function $R(z, \alpha, \beta, f) \in R^{\sigma}_{N+m+n,\alpha}(z, f)$ such that it attains the

$$\min \sum_{k=0}^{qn-1} |f(\beta\omega^k) - R(\beta\omega^k, f)|^2, \qquad \omega^{qn} = 1. \tag{5.2}$$

(P_2^*) If $r_{N+m+n}(z, f) \in R^{\sigma}_{N+m+n}$ minimizes the difference $|f(z) - r_{N+m+n}(z, f)|$ in the L^2-norm on $|z| = 1$, we want to find the regions of equiconvergence of the difference $|r_{N+m+n}(z, f) - R(z, \alpha, f)|$ and $|r_{N+m+n}(z, f) - R(z, \alpha, \beta, f)|$ respectively.

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